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Multidimensional Analysis of Conceptual Understanding of Integer Addition and Subtraction

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Abstract: The aim of this study is to analyze secondary school students' conceptual understanding of integer addition and subtraction. To achieve this, a test based on the multidimensional assessment approach Skills, Properties, Uses, and Representations (SPUR) was administered to 34 secondary school students selected through convenience sampling at a secondary school in Puebla, Mexico. The study employs a qualitative research methodology at a descriptive depth level. The analysis of student responses utilized basic descriptive statistics and the qualitative content analysis technique. Significant differences were found in the accuracy levels across the SPUR dimensions: skills (51.8%), properties (41.9%), uses (60.8%), and representations (20.6%). Students with a solid conceptual understanding of integer addition and subtraction demonstrated high accuracy levels in at least three dimensions. The main difficulties identified were the use of the negative sign and challenges in adding two negative integers. The SPUR approach is a viable alternative for multidimensionally analyzing and assessing conceptual understanding of integer addition and subtraction.

Keywords: *Assessment, conceptual understanding, integers, secondary school, SPUR approach.*

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Introduction

Assessment is an essential component of mathematical learning, whose interest has grown in Mathematics Education research in various countries. This interest arises from the need to support and improve the teaching and learning of abstract concepts in mathematics (Bleiler & Thompson, 2013; Desfitri & Vermana, 2019). Teachers can assess the conceptual understanding of the various aspects that constitute this mathematical ability (Görgüt & Dede, 2020). A multidimensional approach to conceptual understanding proposed by Thompson and Kaur (2011) suggests evaluating mathematical concepts across four dimensions: skills, properties, uses, and representations. Skills refer to fluency in performing procedures and algorithms, properties involve the characteristics of the mathematical concept that are identified and applied generally, uses correspond to applications in various contexts or real-world situations, and representations include drawings, images, graphs, or other visual representations of the concept. Thompson and Kaur (2011) argue that skills, properties, uses, and representations are aspects or dimensions that constitute conceptual understanding and are identified through the acronym SPUR.

Different studies on integers have proposed strategies to improve instruction (Cengiz et al., 2018; Stephan & Akyuz, 2012). In addition, emphasis has been placed on identifying the errors and cognitive obstacles faced by many high school students in basic operations with integers (Bishop et al., 2014; Khalid & Embong, 2020; Makonye & Fakude, 2016). Another line of research has analyzed the reasoning followed by students to solve addition and subtraction problems with integers through tests with a specific assessment approach (Aqazade & Bofferding, 2021; Aqazade et al., 2017). Such an approach defines the knowledge and skills that students are expected to possess in this domain.

For example, the studies by Aqazade et al. (2017), Aqazade and Bofferding (2021) and Cengiz et al. (2018) used tests with exercises on addition, ordering and comparison of integers. In a similar study but focused on identifying errors in

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basic operations (including addition and subtraction) with integers, Khalid and Embong (2020) employed an exercise-based test. Another resource to evaluate exercises of operations with integers were students' notebooks (Makonye & Fakude, 2016). In turn, to explore the use of negative numbers, Ural (2016) posed a couple of written questions to students to find out the most common meaning and use. In summary, the purposes of all these studies did not focus on analyzing the conceptual understanding of addition and subtraction of integers, which could be considered a weakness in the assessment of conceptual understanding. In general, there is a tendency to consider a unidimensional approach to the assessment of addition and subtraction of integers. Mainly the skills dimension, i.e., the application of algorithms to solve exercises has been assumed as the only assessment component.

From a broader perspective of understanding, the assessment of mathematical concepts consists of "identifying whether the student is able to explain the concept, apply properties, develop mathematical representations or images, use the concept, and apply algorithms in problem solving" (Wibowo et al., 2022, p. 31). Therefore, conceptual understanding is a complex process involving multiple dimensions: skills, properties, uses, and representations. The assessment of conceptual understanding based on such an approach is considered multidimensional, since it allows a more complete view of students' cognitive abilities and their true level of development of this mathematical ability.

So far, few studies have addressed an assessment approach to the understanding of integer addition and subtraction that considers the skills, properties, uses, and representations of this mathematical concept (Sercenia et al., 2023; Wibowo et al., 2022). The narrow assessment approach may not adequately represent the complexity of the cognitive processes involved in a deep understanding of the concepts. Thus, the lack of an assessment of conceptual understanding based on the SPUR (skills, properties, uses and representations) approach could limit the accurate diagnosis of students' strengths and weaknesses, as well as the design of effective pedagogical interventions.

The interdependence between assessment and teaching requires continuous monitoring of learning through comprehensive approaches that guide the planning of teaching strategies adapted to the needs of students. By using the SPUR approach as an assessment benchmark, mathematics teachers could benefit from strategies to foster the development of mathematical competencies that are increasingly necessary in today's life. The SPUR approach defines diagnostic and formative assessment through four dimensions and can be a tool for teacher planning. The results obtained from the evaluation under this approach will make it possible to identify the limitations and trends of current teaching in relation to addition and subtraction of integers.

Purpose of the Study

- The aim of the study is to analyze the conceptual understanding of addition and subtraction of integers through the SPUR approach (skills, properties, uses and representations) to identify the level of understanding and difficulties in addition and subtraction of integers, through a test based on that approach.

Literature Review

Conceptual understanding is a fundamental competency in scientific education (Konicek-Moran & Keeley, 2015) and in mathematics education (Cengiz et al., 2018; Sercenia et al., 2023). This competency is achieved when students develop procedures that include executing algorithms and calculations, identifying mathematical properties and rules, applying mathematical concepts to the real world, and representing them through images or graphs (Bleiler & Thompson, 2013). In this regard, some research on assessment in Mathematics Education proposes a multidimensional approach to assess conceptual understanding (Bleiler & Thompson, 2013; Sercenia et al., 2023; Wibowo et al., 2022). From this perspective, multidimensional assessment of mathematical concepts involves evaluating the skills, properties, uses, and representations (SPUR) of the mathematical concept. This approach is open to empirical exploration, and studies addressing it in Mexico are limited (Morante-Rodríguez et al., 2024). Furthermore, one of the fundamental concepts for learning algebra is the addition and subtraction of integers and the appropriate use in problem solving (Elkjaer & Jankvist, 2021; Permata et al., 2019).

In addition, several authors have addressed in their studies components or aspects of the conceptual understanding of addition and subtraction of integers. Among these are noted, the ability in handling algorithms and the proper use of integers in problem solving (Aqazade et al., 2017; Aqazade & Bofferding, 2021; Cengiz et al., 2018; Khalid & Embong, 2019; Sercenia et al., 2023; Ural, 2016; Wibowo et al., 2022), understanding of the properties of integers (Fuadiah et al., 2019; Sercenia et al., 2023) and students' ability to generate mathematical representations (Wibowo et al., 2022). All these aspects are essential and constitute dimensions of the conceptual understanding of addition and subtraction of integers. However, the unidimensional approach adopted in these studies is a narrow view of the conceptual understanding of addition and subtraction of integers. This biased view of reasoning in the development of such mathematical competence limits the accurate diagnosis of students' difficulties and the ability to address them effectively.

As Bleiler and Thompson (2013) state, "Dimensions give an idea of the strength of students' mathematical understanding" (p. 294). However, a few studies have expanded the assessment approach of conceptual understanding of operations with integers. In this regard, Sercenia et al. (2023) assessed arithmetic operations with integers through a test that involved three dimensions: skills, properties, and uses of integers. However, the multidimensional approach adopted was not explicitly stated; rather, the interest was to explore the relationship between conceptual understanding, metacognition, and students' academic performance. Meanwhile, Wibowo et al. (2022) assessed the conceptual understanding of operations with integers through explanations with examples and counterexamples, visual representations, application of properties, and algorithm selection to solve problems.

Although the study by Wibowo et al. (2022) adopted a multidimensional assessment approach, it did not focus exclusively on the addition and subtraction of integers, but also on multiplication and division. By covering several operations (Sercenia et al., 2023; Wibowo et al., 2022), the study did not delve more deeply into those operations that students find more challenging, such as the addition and subtraction of integers. This left open the possibility to explore the conceptual understanding of operations that are considered more difficult for students to perform, especially the subtraction of integers (Makonye & Fakude, 2016; Ural, 2016). In this regard, Bishop et al. (2014) explain that the addition and subtraction of integers trigger contradictions that challenge students' prior notions. For example, students face the idea that addition does not always mean increasing, and subtraction does not always mean decreasing, or the idea that it is possible to subtract more than what one has, such as in 3-5. According to Bishop et al. (2014), these challenges must be addressed, as confronting and resolving these contradictions will help students develop a more solid understanding of these operations. Moreover, it is essential to recognize the negative sign's role in integers' addition and subtraction (Bofferding, 2010).

Regarding to the development of this work with Mexican students, we can justify its relevance because Mexico is among the three countries with the lowest performance in mathematics (Organisation for Economic Co-operation and Development [OECD], 2023). The results of this assessment show a significant setback, with special emphasis on mathematics. This highlights the need to implement new assessment approaches in key concepts of mathematical learning as part of an education that responds to students' needs.

Research Question

The following question guided the research:

- What characteristics of the conceptual understanding of addition and subtraction of integers can be analyzed by using a test based on the skills, properties, uses, and representations approach?

Conceptual Framework

A conceptual understanding of mathematics is a fundamental ability that students must develop. This ability involves recognizing current ideas about the concept, evaluating them, and deciding whether they need to be expanded or modified (Sercenia et al., 2023). Konicek-Moran and Keeley (2015) argue that teachers who focus on developing conceptual understanding in their students use diagnostic and formative assessments to listen to students' alternative conceptions, regardless of the apparent simplicity, mistake, or naivety of their responses. These responses serve as the basis for making changes in the thinking of both teachers and students. Diagnostic and formative assessments provide the necessary information to design, plan, facilitate instruction, and appropriately respond to students' ideas and thoughts. While it is important to assess the achievements made, this should be done while considering the students' current stage of conceptual development (Konicek-Moran & Keeley, 2015). These reasons justify the need to develop a diagnostic and formative assessment of students' conceptual understanding.

In this regard, Thompson and Kaur (2011) suggest that assessment can be approached from a multidimensional perspective to evaluate different aspects of conceptual understanding. This multidimensional perspective, known by the acronym SPUR (S) Skills, (P) Properties, (U) Uses, and (R) Representations, aims to assess "the ability with skills, mathematical properties, uses or applications of mathematics, and representations of concepts" (Thompson & Kaur, 2011, p. 17). Figure 1 illustrates each aspect addressed by this approach.

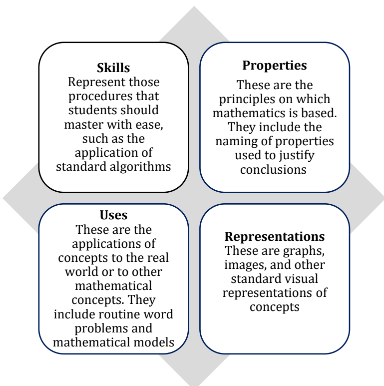


Figure 1. SPUR Approach (Skills, Properties, Uses, and Representations) - Adapted from Thompson and Kaur (2011)

The dimensions grouped in the acronym SPUR (Skills, Properties, Uses, and Representations) serve as indicators for a comprehensive assessment of mathematical concepts or for the design of mathematical tasks with procedural and conceptual character to work in the mathematics classroom (Desfitri & Vermana, 2019; Morante-Rodríguez et al., 2024). This holistic approach accounts for students' abilities to perform computations, abstract properties, apply concepts to different situations, and produce representations of the mathematical concept simultaneously. It is essential to emphasize that assessment of student learning must support teaching without being reduced to routine grading without conscious analysis. Assessment must address the complexity of mathematical thinking (Suurtamm et al., 2016; Thompson & Kaur, 2011). Given the complexity of objectively measuring mathematical concepts, innovative approaches such as the SPUR approach are required.

Methodology

Research Design

In this work, a qualitative approach was used, and it was carried out at a descriptive depth level (Cohen et al., 2007).

Sampling and Data Collection

Convenience sampling was used to select the participants. A total of 34 third-year secondary school students, aged 14 to 15, were selected, including 16 females and 18 males, all from a public school in Puebla, Mexico, with a below-average socioeconomic context. The students have received their mathematics classes in the classroom, with their regular teacher in front of the group, since the beginning of their secondary education. They were designated with the initial "E," followed by an ascending number for identification purposes in this study.

Research Instrument

The study used a test with 21 questions, the researchers developed it based on the SPUR (skills, properties, uses and representations) assessment approach. The purpose of this instrument was to collect data on the conceptual understanding of the addition and subtraction of integers. In this way, the questions explored a specific dimension. It means, questions 1 to 8 explored the skills dimension, questions 9 to 16 explored properties, questions 17 to 19 explored uses, and finally, questions 20 and 21 explored the representations dimension. To ensure the validity and reliability of the instrument, a pilot test was conducted on a group of students from the same grade level, outside the study sample. The authors of this research based themselves on preliminary results to improve the design, clarity, and consistency of the questions. The test used is shown in the appendix.

Data Collection Procedure

For data collection, the researchers obtained permission from the school principal and reported to the group's mathematics teacher. Students were then given a formal permission letter, informing their parents about the confidentiality of the data and the purpose of the study. The permission letter was returned the next day, and the test was administered in the classroom in November 2024. After completing the test in a 45-minute class session, the data collected were subjected to descriptive analysis and qualitative content analysis (Mayring, 2014).

Data analisis

The analysis of this study occurred in two phases. In the first phase, the first two authors assigned scores of 1 and 0 to the students' test responses based on the categories: correct or incorrect, respectively. To examine the drawings in the dimension of representations, subcategories were added (see Table 5). At this point, the third author verified the analysis to ensure that the assigned scores were consistent and resolved any discrepancies between the scores of the first two authors. Together, the researchers conducted a descriptive analysis of the data obtained from the test, with the aim of organizing and presenting them in an orderly manner (Patton, 2014). The numerical results (number of correct answers in each dimension: skills, properties, uses, and representations, and the number of correct answers for each question) guided the analysis and served as the basis for deepening the interpretation of the observed trends. Thus, the study included quantitative data, based on the idea that each qualitative study is unique, which leads to the use of a particularly flexible analytical approach (Miles et al., 2014, as cited in Patton, 2014). The visual representations included tables, figures, and bar charts, which were used to expose the key variables of all the collected data. This exploratory phase helped to analyze the data preliminarily and prepare them for the next stage.

In the second phase, the technique of qualitative content analysis (Mayring, 2014) was applied, considering the deductive categories: skills, properties, uses, representations, and conceptual comprehension of addition and subtraction of integers. This technique allowed for comparisons, identification of difficulties, and determination of which aspects required a deeper analysis. Through this, the main difficulties that were not evident in the descriptive analysis were examined. Additionally, in line with the approach and research question, this analysis was used to interpret the data more thoroughly. As Patton (2014) notes, qualitative analysis seeks to understand a large volume of data by identifying meaningful patterns and constructing a framework that enables a detailed interpretation of that data.

Results

The responses of 34 students were analyzed in the test, which aimed to assess the conceptual understanding of addition and subtraction of integers among the participants. In the descriptive analysis of the data, the number of correct answers from the students in each of the dimensions—skills, properties, uses, and representations—was considered, expressed as percentages. This analysis was carried out by consolidating the responses in a spreadsheet using Excel software. Figure 2 shows the percentage of correct answers from the students in each of the four dimensions of the SPUR approach.

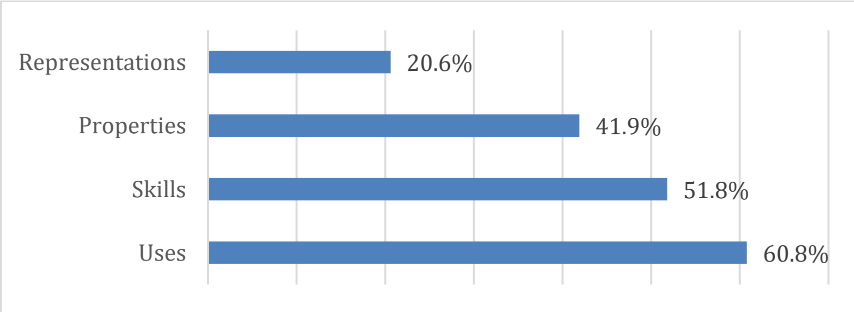


Figure 2. Percentage of Correct Answers by Dimension

The descriptive analysis was carried out for each dimension, meaning that the total number of correct responses to the questions within each dimension was considered. Overall, the highest percentages of correct answers achieved by the students were in the dimensions uses (60.8%) and skills (51.8%) of addition and subtraction of integers. The percentage of correct answers in the properties dimension was moderate (41.9%), while the results showed a much lower percentage of correct answers in the representations dimension, as it was minimal (20.6%).

The number of correct answers for each question within the dimensions of the test was also identified. Table 1 shows the number of correct responses for each of the questions belonging to the different dimensions that made up the test.

Table 1. Correct Answers of Thirty-Four Students for Each Question Included in the Test

Dimensions of the SPUR approach							
Skills		**Properties**		**Uses**		**Representations**	
Question	**Correct answers**	**Question**	**Correct answers**	**Question**	**Correct answers**	**Question**	**Correct answers**
1	26	9	24	17	23	20	3
2	11	10	2	18	19	21	11
3	17	11	26	19	20		
4	32	12	18				
5	14	13	5				
6	10	14	14				
7	16	15	16				
8	17	16	9				

The descriptive analysis revealed that the questions with the lowest number of correct answers (2, 6, 10, 13, 18, 19, 20, 21) represented significant difficulties for the participants, as shown in Table 1. The authors analyzed these difficulties to identify the underlying causes.

Deductive analysis category: skills

In this category, difficulties were classified regarding the subtraction of a positive number b from a negative number a, and the addition of two negative numbers. These difficulties were found to be related to the interpretation of the minus sign in these operations. The analysis was conducted based on the most common incorrect responses to questions 2 and 6 from the Skills dimension. Table 2 shows the questions analyzed in this category and the most frequent incorrect response for each operation.

Table 2. Questions in the Deductive Analysis Category: Skills

Question	Operation	Most common incorrect answers
2	$-7-8=$	-1, 1, 15
6	$-7+(-5)=$	12, 2, -2

In the operation -7 - 8 =, the result -1 indicated the tendency to consider the number with the greatest absolute value and subtract the number with the smallest absolute value, obtaining 1 as the result. The minus sign placed as a unary operator for the result was intuitively placed, as it was observed as a unary sign in both the minuend and the subtrahend. Therefore, students mistakenly interpreted that the result should also be negative. Another explanation is that they considered the binary minus sign (subtraction) as a unary operator for the number with the greatest value and therefore placed it as the result. This means that the students subtracted the number with the smaller absolute value from the number with the greater absolute value, and the minus sign that accompanies the second operand was considered a unary sign, instead of a binary operator indicating the subtraction of a positive number from a negative number.

The result where 1 was obtained indicates that the students considered the higher value and subtracted the lower value, then performed the multiplication of the signs (-) (-) = +, which led them to the result. Another procedure they followed involved adding the absolute values of both numbers and multiplying the two negative signs in the expression, which led to the result of 15.

In the operation -7 + (-5) =, the result 12 indicated that the students identified the plus sign as binary to obtain the sum of both values. However, to assign the unary predicative sign of the result, they multiplied the unary predicative signs of the addends, thus obtaining a positive result. This result indicated the difficulty in operating with the sum of two negative numbers.

The procedure followed by the students to obtain the result of 2 is explained by the subtraction they performed between the number with the greater absolute value and the number with the lesser absolute value, and by the multiplication of the unary signs of the addends. Other students obtained the result of -2. They assumed that -7 should be added to -5, not in the sense of adding two negative numbers (i.e., increasing the number of negative numbers), but in the opposite sense. The students added -5 to -7 as if +5 were being added to -7, moving 5 positions to the right from the position of -7 on the number line.

Deductive category of analysis: properties

In this category, the main difficulties in recognizing the properties of positive integers and the rule for adding a positive integer to a negative integer were grouped, through the analysis of questions 10 and 13 from the properties dimension. Table 3 shows these questions and the various responses from the students.

Table 3. Questions that Caused the most Difficulty in the Properties Dimension

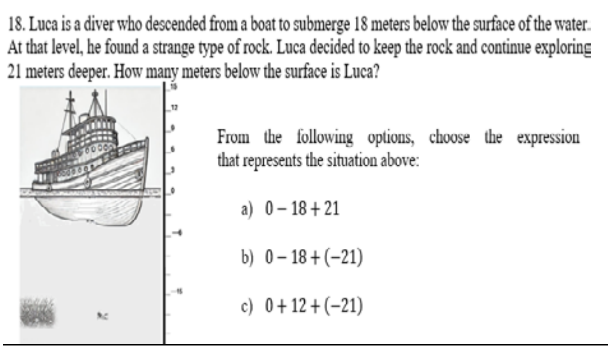
Question 10: And in the case of positive integers, how would you call them?	**Question 13: If you add a positive integer with a negative integer, the result is a number…**
	Most common answers
Blank	Blank
Augmentatives	Negative
Negative	Positive
Positive numbers	Positive-negative
Integers	Positive or negative, depending on the quotient
Integers that no longer carry the sign	Positive, because according to the rules, positive plus negative is positive
Normal numbers	
Infinite	No number
Simple number	It is zero

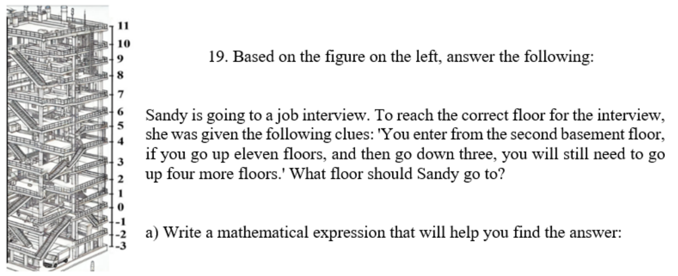
As shown in Table 3, among the most common responses to question 10, none expressed the recognition of positive integers as numbers greater than zero, suggesting that students have a limited understanding of the mathematical characteristics of positive integers. Another notable difficulty among students was the failure to identify the rule for adding any positive integer to any negative integer. The common responses to question 13 revealed that some students incorrectly applied the multiplication rule for signs to the sum of two numbers with different signs. Other students assumed the numbers had the same magnitude, and being of opposite signs, they gave the answers 'zero' and 'no number'; however, the question did not specify this condition.

Deductive category of analysis: uses

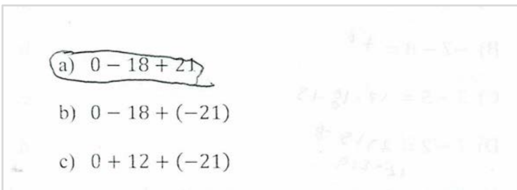
The information concentrated in this category refers to the difficulty in selecting and writing an appropriate mathematical expression that represents the given situation and the solution for each problem. Questions 18 and 19 were analyzed, along with their respective most common incorrect answers. Table 4 presents these questions, the most common incorrect answers from the students, and representative examples of these answers.

Table 4. Questions from the Uses Dimension





Representative examples of incorrect answers



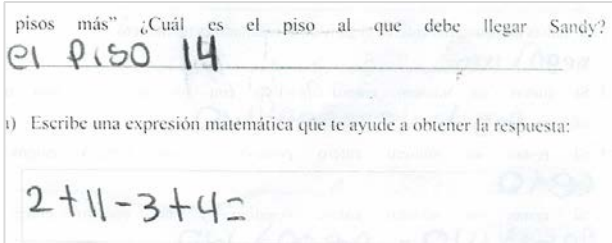


Table 4 shows the most common incorrect responses to questions 18 and 19. Regarding the first response, it can be stated that most students did not identify the use of the sum of two negative numbers to represent the increase in depth relative to sea level, as described in the given scenario. The choice of expression shown in option a represents the diver's initial position plus 21 meters towards the surface, which is incompatible with the information provided in the problem statement.

On the other hand, based on the most common incorrect response to question 19, it was noted that students correctly applied the addition and subtraction of integers, but they did not use the negative sign to represent "the second basement floor" according to the situation described. In other words, students did not associate the idea of something below a given reference level (zero) with the use of the negative sign.

Deductive category of analysis: representations

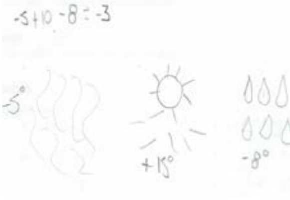
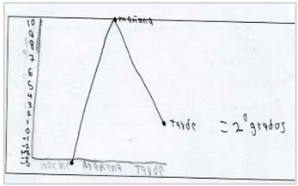
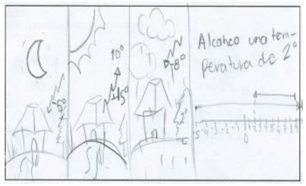
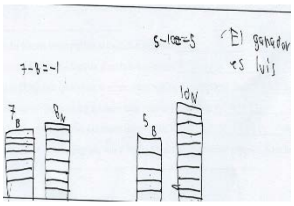
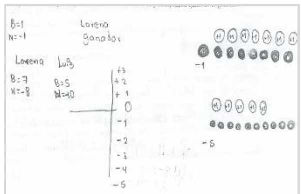
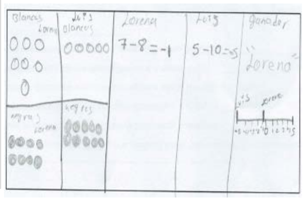
The information addressed in this category refers to the findings on the representation of the addition and subtraction of integers through drawings. The analysis was carried out based on the drawings from questions 20 and 21 made by the students. Table 5 shows the subcategories in which these drawings were grouped, the assigned codes, and corresponding descriptions for each subcategory.

Table 5. Subcategories, Codes, and Description of the Drawings

Subcategory	Code	Description
Pictorial drawings	DP	Representations with an emphasis on the narrative elements of the problem
Schematic drawings	DE	Representations with an emphasis on the mathematical elements of the problem
Pictorial and schematic drawings	DP-E	Representations of the narrative and mathematical elements of the problem
Nonexistent or disconnected drawings	DI	Drawings not made, or not related to the context of the problem

Table 6 shows examples of the subcategories mentioned in the previous table.

Table 6. Drawing Categories for Questions 20 and 21

Questions	Subcategories of drawings		
	DP	DE	DP-E
20. During the night, the temperature dropped to -5 degrees Celsius. In the morning, it increased by 15 degrees. Later, in the afternoon, it rained and the temperature dropped by 8 degrees. Create a drawing that represents the situation described above and determine the temperature reached in the afternoon.			
21. In a game, there are white and black pieces. Each white piece is worth 1 point, and each black piece is worth -1 point. The winner is the one who achieves a score closest to zero. Lorena drew 7 white pieces and 8 black pieces, while Luis drew 5 white pieces and 10 black pieces. Use a drawing to represent the situation and determine who the winner is.			

As shown in Table 6, the examination of the drawings revealed that students are at different stages in mastering the skills to represent narrative or mathematical elements of problems. The students' drawings were classified into subcategories as shown in Table 5. These were based on either narrative or pictorial elements (DP), mathematical or schematic elements (DE), or a combination of both (DP-E). The authors considered as narrative elements the objects, space, and temporal sequence of the story told in the problems. These elements predominated in the DP and were found to be related to incorrect answers. A significant number of students produced DP and DI.

On the other hand, the mathematical elements of a problem are the abstract elements that help solve it and are predominant in the DE, such as quantities and their relationships expressed through symbols, mathematical expressions, and the number line. Identifying both types of elements (narrative and mathematical) from a problem and developing the ability to represent them was shown to be necessary for successful problem-solving. Thus, the drawings in the DE and DP-E subcategories aligned with correct answers, although these were scarce.

Deductive category of analysis: conceptual understanding of addition and subtraction of integers.

In this category, the comparison of the levels reached by the students in the skills dimension and the other dimensions of the SPUR approach was gathered. All students were identified with the initial E and a consecutive number for tracking and comparing the levels they reached.

Figure 3 shows the results of the students with the highest accuracy levels in the skills dimension, as well as their results in the Properties, Uses, and Representations dimensions in the test.

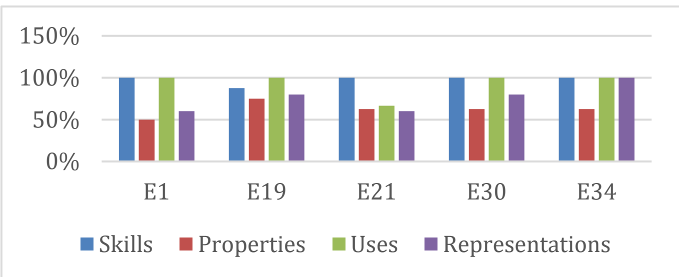


Figure 3. Students with High Accuracy Levels

The students E1, E19, E21, E30, and E34 demonstrated a strong command of the operational skills for adding and subtracting integers, as can be seen in Figure 3. These same students, apart from E21, also showed high accuracy levels in the uses dimension. The accuracy level in the representations dimension was moderately associated with the accuracy levels in skills and uses. As a result, students with high accuracy levels in the skills dimension performed similarly in other dimensions, meaning there was a close relationship in accuracy levels between the skills, uses, and representations dimensions.

On the other hand, the accuracy level in the properties dimension was not related to the accuracy levels in the other dimensions. This suggests a lower development of these students' ability to clearly identify the properties or rules to follow in the addition and subtraction of integers. These results indicate that students E19, E30, and E34 have a solid conceptual understanding of the addition and subtraction of integers

The following presents the results of the students with a moderate accuracy level (50%-75% correct answers) in the skills dimension. These results and the comparison with the Properties, Uses, and Representations dimensions are shown in Figure 4.

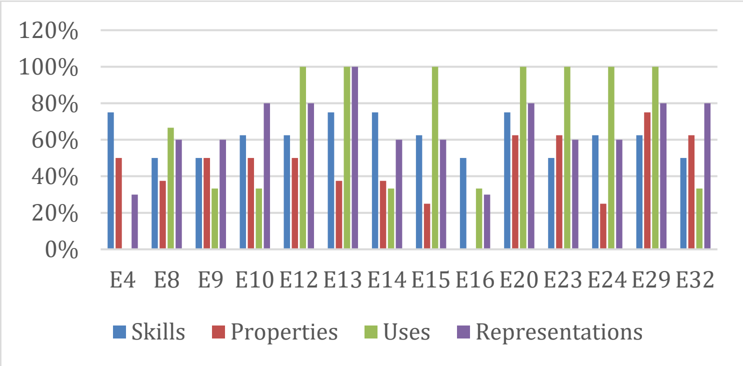


Figure 4. Students with Moderate Accuracy Levels

Based on the results shown in Figure 4 regarding the students E8, E9, E10, E12, E13, E14, E15, E20, E23, E24, E29, and E32, it can be observed that there is a relationship between the students' accuracy levels in the skills and representations dimensions. Regarding the uses and properties dimensions, more pronounced differences are shown between the students presented in Figure 4. Therefore, these results suggest an acceptable level of conceptual understanding.

Finally, in Figure 5, the students with a low accuracy level (below 40%) in the skills dimension and their results in the Properties, Uses, and Representations dimensions are shown.

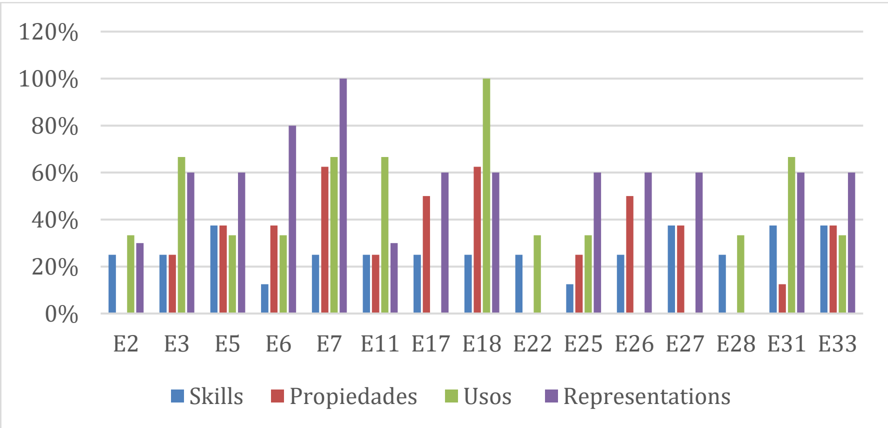


Figure 5. Students with Low Accuracy Levels

As shown in Figure 5, the accuracy level in the skills dimension for students E2, E3, E5, E6, E7, E11, E18, E22, E25, E26, E27, E28, E31, and E33 was very low, as well as in the other dimensions, except for a few cases where accuracy levels were high (E7 and E18). The accuracy levels in the skills and properties dimensions did not exceed 60%, except for E7. These results suggest that these students have a superficial conceptual understanding of the addition and subtraction of integers.

Discussion

The results obtained from the descriptive analysis showed notable differences between the accuracy levels in the skills, properties, uses, and representations dimensions. Students showed greater fluency in applying addition and subtraction of integers to real-world context problems (60.8%) and in calculation and algorithmic procedures (51.8%). In the properties dimension, mastery was below 50%, while a significantly lower mastery was observed in creating representations of situations involving the addition and subtraction of integers (20.6%). This finding may be related to the tendency of mathematics teachers to evaluate only one dimension of the SPUR approach, primarily the skill dimension, and their efforts to address the uses dimension in their teaching and assessments (Görgüt & Dede, 2020). Furthermore, this result is also consistent with previous studies by Khalid and Embong (2020) and Wibowo et al. (2022), who found that students' lack of skill in generating mathematical representations is the cause of difficulties in solving problems involving operations with integers.

After the descriptive analysis, and for a deeper interpretation of the data, the results of the qualitative content analysis focused on the main difficulties identified in questions 2 and 6 (category skills), 10 and 13 (category properties), 18 and 19 (category uses), and 20 and 21 (category representations).

In the operations -7-8= and -7+(-5) = (questions 2 and 6 of the skills category), a trend was observed where students subtracted the number with the smaller absolute value from the number with the larger absolute value to solve the operation. In some cases, they added the two absolute values; however, they performed the multiplication of the negative signs in the expression. This was incorrect because it involved subtracting a positive number, *b*, from a negative number, *a,* and adding two negative numbers rather than performing multiplication. Based on these results, two points can be made. The first is that students did not clearly identify the role of the negative sign. This result confirms that the meaning students assign to the negative sign, either as a unary sign (negative sign accompanying a number) or binary sign (indicating the operation of subtraction), is a crucial element in operations with integers, as demonstrated in previous studies (Bofferding, 2010; Elkjaer & Jankvist, 2021; Vlassis, 2008). The second point is that the presence of parentheses did not influence the resolution of the operations since students multiplied the negative signs in the expression in both operations with and without parentheses.

In questions 10 and 13 (from the properties category), students presented unclear ideas about the characteristics of positive integers. This finding contrasts with the results found by Cengiz et al. (2018), who discovered that younger students have a basic intuitive understanding of integers that allows them to give meaning to the addition and subtraction of integers.

Additionally, it was identified that the rule for adding a positive number to a negative number is often confusing for students. This may be because students hold on to ideas about the addition and subtraction of natural numbers, which persist and are difficult to change. For example, the idea that addition always involves an increase and subtraction always involves a decrease. Furthermore, students need to understand that they can subtract a larger number from a smaller one, as in 3 - 5, which is not always intuitive (Aqazade et al., 2017; Bishop et al., 2014). They also need to learn to view negative numbers abstractly, as concepts—beyond physical or concrete situations. When students fail to develop this capacity for abstraction, they find it difficult to generalize the behavior of negative numbers, which prevents them from correctly applying the rule for adding a positive number to a negative one.

In question 18 (from the uses category), the results showed that several students had difficulties choosing an appropriate mathematical expression to represent the problem. Another study in a similar line has confirmed that difficulties also arise when, starting from a mathematical expression (the sum of integers), most students fail to formulate an appropriate problem (Ural, 2016), or create representations from the problem (Wibowo et al., 2022). In this same category, the results from question 19 reflected that most students did not associate the use of negative numbers with the context of the building shown in the problem, where they were supposed to assign the negative sign to the second floor of the basement. The limited use of negative numbers in various contexts was also observed in U.S. students, who often alter mathematical expressions or create stories with unrealistic scenarios (Wessman-Enzinger & Mooney, 2021). This indicates the need to broaden the contexts to teach the addition and subtraction of integers.

In summary, the practical contexts (such as depth below sea level and underground floors) and the vertical number line in the images that accompanied problems 18 and 19 seemed to have no impact on students in solving integer addition and subtraction problems. This could indicate that, when the understanding of the concept of integer addition and subtraction is superficial, practical contexts lose relevance, and students are unable to use the representations. These results might be related to the traditional teaching approach, which emphasizes solving exercises without addressing the underlying processes in problem-solving. Therefore, students' ability to solve problems and apply their knowledge to everyday situations is lower than expected (OECD, 2023). These findings highlight the need to improve the teaching of problem-solving with integers, especially regarding the translation between the statement, the mathematical expression, and the use of representations (Makonye & Fakude, 2016).

The correct answers to questions 20 and 21 (from the representations category) were only the drawings accompanied by the vertical number line to represent temperature variations and the relationships of increase or cancellation in the chip game, thus identifying the sign of the result and its distance from zero. It was identified that the use of the vertical number line helped students see negative numbers abstractly. That is, as positions on the number line and movements in the direction indicated by the integers. This result is in line with studies that support the use of the number line as an effective teaching and learning strategy for integer addition and subtraction (Bishop et al., 2014; Cunningham, 2009).

Thus, this study confirmed that the use of the number line is a cognitive possibility, that is, a form of reasoning that can facilitate and improve the understanding and work with integers, especially negative ones (Bishop et al., 2014). In this regard, Aqazade et al. (2017) pointed out that the addition of integers from the position of the negative number makes it easier to add the second number upwards or downwards and helps in understanding the minus sign as a unary symbol. Furthermore, our result regarding the identification of the token game context to promote the use of the number line aligns with the previous work of Bofferding and Wessman-Enzinger (2017), who suggested that concrete models, such as the token model, could help the number line emerge naturally from reasoning, rather than being imposed when solving subtraction problems with integers.

Finally, it is important to note that the results of this study align with what Konicek-Moran and Keeley (2015) stated. These authors argue that understanding a concept is evident when students can think about, use, express, and represent that mathematical concept. This ability was manifested at different levels of development, as demonstrated in this study. The SPUR approach allowed for the examination of the various dimensions involved in conceptual understanding, the analysis of difficulties that arise, and the evaluation of possible improvements in teaching and learning, as addressed in previous studies (Bishop et al., 2014; Sercenia et al., 2023; Wibowo et al., 2022).

Conclusion

The study revealed key aspects in the conceptual understanding of integer addition and subtraction. It has been shown that the SPUR approach allows for assessing students' level of conceptual understanding, as well as analyzing the difficulties most students face in integer addition and subtraction. In this sense, it is recognized that students should not only know the basic rules of these operations and solve exercises but also be able to apply these concepts and their representations to solve problems in various contexts. The findings suggest that the use and representation dimensions can be effectively explored through problem-solving in contexts such as: depths underwater, floors below ground, temperatures, and chip games. Additionally, it has been identified that the vertical number line is a key reasoning strategy for conceptualizing negative numbers abstractly, treating them as points on the number line. This could facilitate understanding the minus sign as a unary operator and improve the understanding of integer addition and subtraction.

This study reinforces the idea that assessment through the SPUR approach links the analysis of exercises, rules, problems, and visual representations that students must learn, generate, and solve, which constitutes a deep and flexible understanding of integer addition and subtraction. These findings offer valuable implications for designing teaching strategies that promote a stronger and more conceptual understanding of integers.

Recommendations

Based on the results and conclusions obtained, the authors suggest:

It is crucial for educational authorities and mathematics teachers to become familiar with the SPUR approach through courses and workshops to strengthen their teaching efforts. By implementing this approach, educational authorities can gain valuable insights, such as identifying limitations in teaching and trends that hinder learning, as well as managing the continuous professional development of mathematics teachers to incorporate the design of tasks and assessments based on this comprehensive approach. Mathematics teachers could use the SPUR assessment approach as a reference for diagnostic and formative evaluation. The results of assessments under this comprehensive approach can guide the planning of teaching strategies tailored to the specific needs of students, either before or during instruction, and prevent delays in learning more advanced mathematical concepts, such as those encountered in algebra.

Through this approach, mathematics teachers can improve teaching by emphasizing the dimensions where students face difficulties, as well as reinforcing the areas where they already show mastery, generating the recognition and encouragement needed for them to progress toward a deeper understanding of mathematical concepts. It is essential for the SPUR approach to also be used as a tool for designing tasks, both for the mathematics classroom and for textbooks, in a way that promotes a comprehensive teaching of mathematical concepts and aligns with assessment, as you cannot assess what is not taught.

Future studies should examine in more detail the difficulties students face when using the rule for adding integers and explore the role of parentheses in such expressions. It is also suggested to continue exploring the use of practical contexts, such as the chip model, in teaching problem-solving with integers, as well as in the translation between the statement, the mathematical expression, and the representations. Additionally, it is recommended that future studies improve convenience sampling by increasing the number of participants and strengthening the statistical validation of the test to control bias in the results. Finally, future research could evaluate other mathematical concepts through the SPUR approach.

Limitations

One of the limitations of this study is the lack of follow-up through in-depth interviews, which would have allowed for capturing more information from students to understand integer addition and subtraction, as has been done in other studies. Another limitation is the representativeness and generalization of the findings, as convenience sampling was used. However, this sampling technique offered several advantages. On the one hand, it facilitated access to participants without using methods that require more time and elaborate procedures. On the other hand, it allowed for the collection of a large amount of qualitative data to better comprehension the characteristics of students' understanding of integer addition and subtraction. Another relevant aspect is that the test could be statistically validated to improve the representativeness and generalization of the findings.

Ethics statements

The researchers took essential ethical principles into account when conducting this study. Before beginning data collection, the purpose of the study was explained to the participants, and both they and their parents gave informed consent. Additionally, data support was reported for the sole purpose of this study.

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Conflict of interest

There is no conflict of interest in this study

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Authorship Contribution Statement

Toxtle-Colotl: Conceptualization, data collection, design, analysis, and writing. Nieto-Ruiz: Conceptualization, data collection, and writing. Juárez-López: Review and supervision

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Appendix

Addition and subtraction of integers comprehension test

Please write the following information:

Grade: ________ Group: ______ Date: _______

Answer what is indicated in each section. Each one is designed to test a different aspect of your understanding of adding and subtracting negative numbers.

1. **Skills**

Solve the following operations with negative numbers:

2. **Properties**

Read the following and answer:

9. If a number is less than zero, is it equivalent to saying that it is a negative number? _________

A) $-10 + 15 =$

B) $-7 - 8 =$

C) $3 - 5 =$

D) $7 - 2 =$

a. $8 + (-3) =$

b. $-7 + (-5) =$

c. $12 - (-6) =$

d. $-6 - (-5) =$

Explain your answer

____

____

10. And in the case of positive integers, what would you call them? ____

11. If you add two positive integers, the result is a number

____

12. If you add two negative integers, the result is a number

____

13. If you add a positive integer to a negative integer, the result is a number

____

14. If you subtract a positive integer and a negative integer, the result is

____

15. If you subtract a negative integer and a positive integer, the result is

____

16. If you add two integers of the same value or magnitude, but opposite or opposite signs, the result is

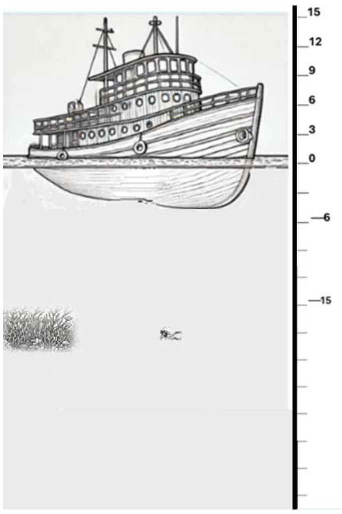
____

3. Uses

Solve the following problems.

17. The temperature was -4 degrees tonight; since then, it has dropped 9 degrees. What is the temperature now?

18. Luca is a diver who descended from a boat to dive 18 m below the surface of the water. At that level he found a kind of strange rock. Luca decided to save the rock and continue exploring 21 meters deeper. How many meters will Luca be above the surface?

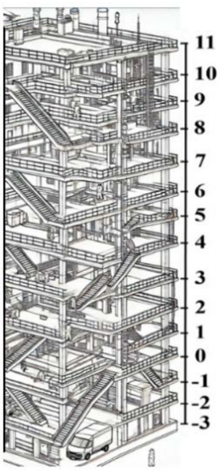


From the following options choose the expression that represents the previous situation:

a) $0 - 18 + 21$

b) $0 - 18 + (-21)$

c) $0 + 12 + (-21)$



19. Based on the figure on the left, answer the following:

Sandy goes to a job interview. In order for him to get to the correct floor for the interview, they only gave him the following clues: “You enter through the second floor of the basement, if you go up eleven floors, and then go down 3, you will have to go up four more floors” What is the floor to which Should Sandy arrive?

a) Write a mathematical expression that helps you get the answer:

4. Representations

20. During the night the temperature dropped to -5 degrees Celsius. In the morning, it rose 15 degrees. Then, during the afternoon, it rained and the temperature dropped 8 degrees.

Make a drawing that represents the situation described above and determine what temperature was reached in the afternoon.

21. In a game you have white pieces and black pieces. Each white tile is worth 1 point and each black tile is worth -1 point. Whoever achieves a score that is closest to zero wins. Lorena took out 7 white pieces and 8 black pieces and Luis took out 5 white pieces and 10 black pieces.

Use a drawing to represent the previous situation and check who is the winner.